Parameterization for fitting triangular mesh*

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Abstract In recent years, with the development of 3D data acquisition equipments, the study on reverse engineering has become more and more important. However, the existing methods for parameterization can hardly ensure that the parametric domain is rectangular, and the parametric curve grid is regular. In order to overcome these limitations, we present a novel method for parameterization of triangular meshes in this paper. The basic idea is twofold: first, because the isotherms in the steady temperature do not intersect with each other, and are distributed uniformly, no singularity (fold-over) exists in the parameterization; second, a 3D harmonic equation is solved by the finite element method to obtain the steady temperature field on a 2D triangular mesh surface with four boundaries. Therefore, our proposed method avoids the embarrassment that it is impossible to solve the 2D quasi-harmonic equation on the 2D triangular mesh without the parametric values at mesh vertices. Furthermore, the isotherms on the temperature field are taken as a set of iso-parametric curves on the triangular mesh surface. The other set of iso-parametric curves can be obtained by connecting the points with the same chord-length on the isotherms sequentially. The obtained parametric curve grid is regular, and distributed uniformly, and can map the triangular mesh surface to the unit square domain with boundaries of mesh surface to boundaries of parametric domain, which ensures that the triangular mesh surface or point cloud can be fitted with the NURBS surface.

Keywords: parameterization, regular parametric curve grid, mesh fitting, reverse engineering.

With the development of 3D acquisition equipments, parameterization of triangular mesh or point cloud is becoming an important problem in reverse engineering of shape design, because reconstructing free-form surface from point cloud, or fitting triangular mesh with four boundaries with NURBS surface, requires the parameterization of the point cloud or mesh vertices in advance, that is, seeking a set of regular parametric curve grids with distribution as uniform as possible.

Currently, one of the main approaches to parameterization is to calculate the harmonic field on a triangular mesh. Eck et al. calculated the pseudo-harmonic field on a triangular mesh surface by a linear map approximating the harmonic map[1]. Pinkall and Polthier proposed a discrete harmonic map[2]. However, parameterization with either of the above methods cannot promise to avoid foldovers, because some weights in the map may be negative[3,4]. Therefore, they are not effective for all triangular meshes. Recently, based on the mean value theorem for the harmonic function, Floater presented a Mean Value Coordinates method, that is, searching a map approximating the harmonic map in a set of convex linear combination maps[4]. Unfortunately, the obtained pseudo-harmonic field did not converge to the true harmonic field. In fact, calculating the harmonic field on a triangular mesh surface is equivalent to solving a boundary value problem of the 2D harmonic equation. However, without the parameterization of the triangular mesh, it is impossible to solve the boundary value problem with either the finite difference method or the finite element method.

To overcome these limitations and parameterize triangular mesh, two key problems must be solved first: the fold-over in the parameterization, and the initial parameterization of mesh vertices for solving the 2D harmonic equation. In this paper, we present a novel method which can solve the above two key problems, thereby generating ideal parameterization on triangular meshes.

The idea for solving the first key problem exploits the non-intersection and uniform distribution of the isotherms in the steady temperature field. Taking the 2D case as an example: after constructing the steady temperature field on a mesh surface with four boundaries, which is determined by a boundary prob-

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lem of the 2D harmonic equation[5] with the temperature values at one pair of boundaries assigned to be 0 or 1, and the other pair from 0 to 1, the temperature value at each mesh vertex can be taken as its parametric value. Interestingly, since the temperature value at each point of the steady temperature field is unique, the isotherms in it neither intersect nor self-intersect. Therefore, it ensures that there is no fold-over in the parametric domain, if taking the isotherms as the iso-parametric curves on the mesh surface.

To conquer the second key problem, the 3D harmonic equation is solved by the finite element method. Usually, the harmonic field on a triangular mesh is obtained by solving a boundary problem of the 2D harmonic equation. However, as mentioned above, without the initial parametric values at mesh vertices, it is impossible to solve the 2D boundary problem using either the finite difference method or the finite element method. In this paper, we propose a novel method, by which we get the steady temperature field (quasi-harmonic field) defined on the triangular mesh surface. Specifically, we calculate an offset surface of the original triangular mesh surface firstly, and then, construct a lamina taking the two surfaces as its border surfaces. After giving the boundary condition of its four side surfaces, the harmonic field (steady temperature field) on the lamina can be obtained by solving the boundary value problem of the 3D harmonic equation with the finite element method. At the same time, we also get the steady temperature field restricting on the original triangular mesh surface, which satisfies a quasi-harmonic equation, and is a quasi-harmonic field. Our method has the following three advantages over the existing methods: 1) The method is effective to any triangular mesh with four boundaries; 2) the so obtained steady temperature field on the triangle mesh surface is quasi-harmonic field, in which the isotherms can be taken as a set of iso-parametric curves on the triangular mesh surface; 3) the other set of iso-parametric curves can be generated by connecting the points with the same chord length in the isotherms. The two sets of iso-parametric curves construct a regular parametric curve grid, and the triangular mesh surface can be parameterized into the unit square by the regular parametric curve grid. Due to the property of the harmonic equation, the parametric curve grid is distributed uniformly and approaches the orthogonal grid; and more importantly, there is no fold-over in the so obtained parametric domain. Many experiments in this paper demonstrate the robustness and applicability of our approach.

1 Related work

Many researchers have addressed the problem of computing low-distortion parameterizations for triangular meshes. Bennis et al. proposed a piecewise flattening method for free-form parametric surfaces[6]. Floater embeds an open triangular mesh in the plane by mapping its boundary vertices onto a 2D convex polygon and solving a linear system to determine the 2D embedded positions of interior vertices[7]. The linear system is constructed by representing each interior vertex as a convex combination of its neighbor vertices. The shape preserving parameterization can be obtained by using conformal mapping and barycentric coordinates to determine the combination coefficients[8, 9]. Lévy and Mallet extended Floater's approach by defining a set of non-linear constraints on the mapping that ensure local orthogonality[10]. The MIPS method[11, 12] attempts to roughly preserve the ratio of singular values over the parameterization. Haker et al. introduced a method to compute a global conformal mapping from a genus zero surface to a sphere[13]. A recent work by Zigelman analytically found an embedding of an open mesh in the plane by a multi-dimensional scaling (MDS) method that optimally preserves the geodesic distances between mesh vertices[14]. Gu et al. computed a geometric-stretch parameterization using the hierarchical optimization algorithm, which minimizes the geometric stretch[15]. The above parameterization methods strive to minimize the distortion of the triangles, and so the distortion of the texture mapped to the triangular mesh surface is little. Therefore, these parameterization methods are more suitable to texture map than fitting, for good fitting to the triangular mesh needs the regular parametric curve grid with uniform distribution.

2 Preliminaries

2.1 Definitions

Definition 1. Given a parametric surface $\boldsymbol{r}=\boldsymbol{r}(u, v)$, a point $\boldsymbol{r}(u_0, v_0)$ is called a regular point, if $\boldsymbol{r}_u \times \boldsymbol{r}_v|_{(u_0, v_0)} \neq 0$, where $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are the derivatives of $\boldsymbol{r}(u, v)$ with respect to u and v, respectively. If all points on the surface $\boldsymbol{r}=\boldsymbol{r}(u, v)$ are regular points, the parametric curve grid on the surface is called a regular parametric curve grid.

Definition 2. Given a directed curve C_{AB} from point A to point B, the chord length function $L(P)(P\in C_{AB})$ is defined as the chord length from point A to point P. And $L(P)/L(B)$ is called the normalized chord length for a point $P\in C_{AB}$.

Definition 3. Suppose that a heat conductor V with the interior G and the boundary B is composed of continuous medium, the temperature field of V is called a steady temperature field when the heat conduction process within V reaches an equilibrium state.

Based on the principle in heat transfer, the steady temperature field satisfies the following quasi-harmonic equation[5]:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(p(x,y,z)\dfrac{\partial T}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(p(x,y,z)\dfrac{\partial T}{\partial y}\right) \\ \quad +\dfrac{\partial}{\partial z}\left(p(x,y,z)\dfrac{\partial T}{\partial z}\right) \\ \quad = f(x,y,z),\ (x,y)\in G \\ T\ |_B=\varphi(x,y,z) \end{cases}, \tag{1}$$

where $\varphi(x,y,z)$ is the given temperature on the boundary B, $p(x,y,z)$ is the heat conductivity coefficient of the heat conductor V dependent on the physical property of V, and $f(x,y,z)$ is the heat source inside V. For a heat conductor V composed of a pure material, $p(x,y,z)$ is constant for every point inside V. Moreover, if there is no heat source inside V, i.e. $f(x,y,z)=0$, Eq. (1) becomes the following harmonic equation

$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}+\dfrac{\partial^2 T}{\partial z^2}=0,\ (x,y)\in G \\ T\ |_B=\varphi(x,y,z) \end{cases}. \tag{2}$$

By the variational principle[16], Eq. (2) is equivalent to the following variational problem: find the function in the set $M=\{T(x,y,z)\mid T\in C^2(V),\ T|_B=\varphi(x,y,z)\}$ that minimizes the functional

$$I(T)=\iiint_V\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2+\left(\frac{\partial T}{\partial z}\right)^2\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{3}$$

3 Parameterization based on steady temperature field

Given an open oriented triangular mesh surface S with four pieces of boundaries, we want to calculate a set of regular parametric curve grids with distribution as uniform as possible on the surface, and map the vertices of the triangular mesh to the unit square parameter field.

As the triangular mesh S is a 2D manifold, in essence, the steady temperature field (quasi-harmonic field) on S satisfies the 2D quasi-harmonic equation

$$\begin{cases} \dfrac{\partial}{\partial u}\left(p(u,v)\dfrac{\partial T}{\partial u}\right)+\dfrac{\partial}{\partial v}\left(p(u,v)\dfrac{\partial T}{\partial v}\right)=0, \\ \quad (u,v)\in G \\ T\ |_B=\varphi(u,v) \end{cases}. \tag{4}$$

However, the two independent variables (u,v) in the 2D equation are the parameters which we want to calculate for the mesh S.

In fact, without the 2D parameters of the mesh, it is impossible to solve the 2D quasi-harmonic equation (4) on the triangular mesh surface by either the finite difference method or the finite element method. First, discretizing the 2D quasi-harmonic equation with the finite difference method needs the 2D parameters. For example, it was based on an initial parameterization that the method in Ref. [17] solved the 2D harmonic equation using the finite difference scheme. However, their initial parameterization method is not suitable to the vertices of the triangular mesh in 3D. Second, if the boundary value problem of the 2D quasi-harmonic equation (4) is solved using the finite element method, the variational problem must be converted to: finding the function in the set $M=\{T(u,v)\mid T\in C^2(V),\ T|_B=\varphi(u,v)\}$ that minimizes the functional

$$I(T)=\iint_V\left[\left(p(u,v)\frac{\partial T}{\partial u}\right)^2+\left(p(u,v)\frac{\partial T}{\partial v}\right)^2\right]\mathrm{d}u\mathrm{d}v. \tag{5}$$

Accordingly, step 2) of the finite element method in the following Section 3.3 is converted to constructing a linear interpolation function

$$T^{(k)}(u,v)=\lambda_i(u,v)T_i+\lambda_j(u,v)T_j+\lambda_k(u,v)T_k \tag{6}$$

in each triangle. However, without the 2D parameters of the mesh vertices, we cannot get the barycentric coordinates $\lambda_i(u,v)$, $\lambda_j(u,v)$ and $\lambda_k(u,v)$ of the points in the triangle, which are represented with the 2D parameters (u,v). Therefore, the following steps of the finite element method described in Section 3.3 cannot proceed.

In this work, we present a novel method for directly calculating the steady temperature field (quasi-harmonic field) on an arbitrarily given triangular mesh surface S. We first calculate the offset surface U of the triangular mesh S. The surface U is created by displacing each vertex of S and is also a triangular mesh with the same vertex connectivity as S. A lamina V composed of triangular prisms is then generated by connecting the corresponding vertices of S and U (Fig. 1). The lamina V becomes a simplicial complex composed of tetrahedra by dividing each triangular prism into 3 tetrahedra. By solving the 3D harmonic equation (2) on the simplicial complex V, we can get the steady temperature field on V, and accordingly the steady temperature field on its bottom surface S. Thus, the isotherms can be taken as a set of iso-parametric curves on the surface S. Based on Ref. [5], the steady temperature field on the mesh surface S satisfies a 2D quasi-harmonic equation, and is a quasi-harmonic field. However, different distances between the surface S and its offset surface U lead to different steady temperature fields on the surface S, which satisfies different quasi-harmonic equations. The shorter the distance is, the closer the shape of the surface U is to the shape of S, and then, the smoother the isotherms in the steady temperature field on S are, and the more uniform their distribution is.

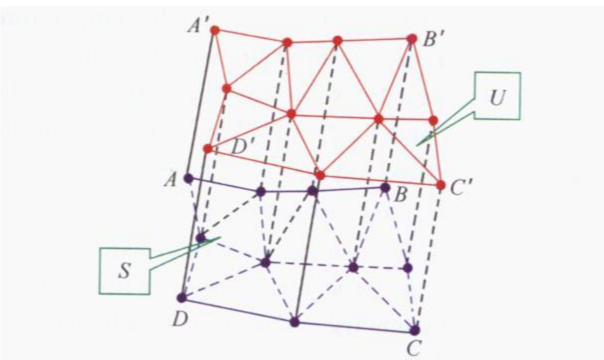

Fig. 1. A triangular prism generated by connecting the corresponding vertices of the surface S and its offset surface U.

In the following sections, we will elaborate the above method.

3.1 Three-dimensional simplicial complex construction

As S is a triangular mesh, its offset surface U with offset distance d can be easily calculated by displacing the vertices along their normal directions of S[18]. The offset surface U is also a triangular mesh with the same connectivity as S. In order to avoid self-intersection of U, the offset distance d is usually chosen to be significantly smaller than the minimum of the absolute value of the reciprocal of the two principal curvatures at the vertices of the surface S[19]. The two principle curvatures at vertex v of surface S are computed by the methods presented in Refs. [20, 21], that is, $k_{1,2} = H \pm \sqrt{H^2 - K}$ with

$$K = \frac{2\pi - \sum_{i=1}^{n} \alpha_i}{\frac{1}{3}A}, \quad H = \frac{\frac{1}{4}\sum_{i=1}^{n} \| e_i \| \beta_i}{\frac{1}{3}A}. \quad (7)$$

Here, n is the number of the triangles adjacent to the vertex v, α_i is the radian of the angle of the i-th triangle adjacent to the vertex v, which takes the point v as its vertex, and A is the sum of the area of all triangles adjacent to the vertex v. e_i is the i-th edge adjacent to the vertex v, $\| e_i \|$ denotes the length of the edge, and β_i is the dihedral angle corresponding to the edge.

Note that the shorter the offset distance is, the closer the shape of U is to the shape of S. It makes the distribution of the isotherms of the steady temperature field on the surface S more uniform.

It should be pointed out that, when the distance between the two surfaces S and U tends to zero, the steady temperature field on the bottom surface S of the lamina V converges to the harmonic field, i.e. the solution to the 2D harmonic equation. In fact, the lamina V changes with the distance between the two surfaces S and U. Therefore, the domain of the 3D harmonic equation varies, leading to different steady temperature fields in it, as well as different isotherm surfaces, and different isotherm curves on the surface S. However, although the distance between the two surfaces S and U is smaller and smaller, a steady temperature field in the lamina V, as well as that on the bottom surface S, can always be calculated by solving the 3D harmonic equation (2). That is to say, when the lamina V tends to its bottom surface S, what is restricted on the bottom surface S is always a steady temperature field, which is determined by the steady temperature field in the varying lamina V. In particular, as the limitation, when the lamina V becomes the surface S, the field on the surface S is still a steady temperature field, which is governed by the 2D quasi-harmonic equation (4). Note that it has been supposed that the heat conductivity coefficient $p(x, y, z)$ is a constant, so the 2 D quasi - harmonic equation (4) is just the 2 D harmonic equation. Therefore, the field on the surface S is just the harmonic field.

The shape difference between S and U can be measured by the maximum of the differences of the Gauss curvatures between the corresponding vertices defined by

$$d_s = \max\{ | K_i^S - K_i^U | \mid i = 1, 2, \cdots, m \}, \quad (8)$$

where K_i^S denotes the Gauss curvature of the i-th vertex of the surface S, and K_i^U denotes the Gauss curvature of the i-th vertex of the surface U. Here, we calculate the Gauss curvature at each vertex by the method presented in Refs. [20, 21]. In order to calculate the offset surface whose shape is sufficiently close to that of the original mesh surface, the following Algorithm 1 is presented for constructing the offset surface. In the algorithm, $k_{\min}$ denotes the minimum of the absolute values of the reciprocal of the two principal curvatures at the vertices of the surface S, ε is the given threshold of the shape difference between S and U, d_o the offset distance, and d_s the shape difference between them.

Algorithm 1. Constructing the offset surface U of the surface S

$d_o = k_{\min}$;
do{
$d_o = d_o/2$;
Construct the offset surface U for the surface S with distance d_o;
calculate the shape difference d_s between the surfaces S and U;
}while($d_s > \varepsilon$)

For each triangle of S, there is a corresponding triangle on its offset surface U. Connecting the three pairs of the corresponding vertices between a pair of corresponding triangles, we can get a triangular prism. All the triangular prisms form a volume V. Whereafter, by dividing each triangular prism to three tetrahedra, a simplicial complex composed of tetrahedra can be obtained[22] (Fig. 1).

3.2 Regular parametric curve grid generation

In Section 3.1, a 3D simplicial complex V composed of tetrahedral is constructed, which takes the surfaces S and U as its border surfaces (Fig. 1). The four pieces of boundaries of the surface S are denoted by C_{AB}, C_{BC}, C_{DC}, and C_{AD}. The opposite boundary to C_{AB} is C_{DC}, and the opposite boundary to C_{BC} is C_{AD}. The offset surface U also has four corresponding boundaries, $C_{A'B'}$, $C_{B'C'}$, $C_{A'D'}$, and $C_{A'D'}$. In order to solve the harmonic equation (2) on the volume V, the boundary condition at the four side faces of volume V should be specified at first. Let the temperature values at the vertices on the boundaries C_{AB} and $C_{A'B'}$ be 0, and the temperature values at the vertices on the boundaries C_{DC} and $C_{D'C'}$ be 1; the temperature values at the vertices on the boundaries C_{BC} and $C_{B'C'}$ be their normalized chord lengths, and the temperature values at the vertices on the boundaries C_{AD} and $C_{A'D'}$ be also their normalized chord lengths.

After giving the boundary condition, the steady temperature field on the simplicial complex V can be solved using the finite element method, which will be described in detail in Section 3.3. Meanwhile, the steady temperature field on the bottom surface S is also obtained. In the steady temperature field of V, there are a series of isothermal surfaces. On the other hand, the bottom surface S can be regarded as the section of the set of isothermal surfaces, and their intersection curves are a series of isotherms on the steady temperature field on the surface S, which can be taken as a series of iso-parametric curves on the surface S.

On the other hand, by assigning the temperature values at the vertices on the boundaries C_{AD} and $C_{A'D'}$ to 0, the temperature values at the vertices on the boundaries C_{BC} and $C_{B'C'}$ to 1, and the temperature values at the vertices on the boundaries C_{AB}, $C_{A'B'}$, C_{DC} and $C_{D'C'}$ to their normalized chord lengths, another set of isotherms on the surface S can be calculated as another iso-parametric curve on the surface. However, the parametric curve grid so obtained does not guarantee the regularity.

In order to obtain the regular parametric curve grid, after getting a set of iso-parametric curves on the surface S, the normalized chord length at each point in each iso-parametric curve is calculated as another parameter value of the point. Specifically, because the surface S is a piece of triangular mesh surface, we first calculate the iso-parametric curve through each vertex of the surface, and then, calculate the normalized chord length at the vertex in the iso-parametric curve as its another parameter value. Thus, another set of iso-parametric curves on the surface S are constructed. Obviously, the parametric curve grid composed of the two sets of iso-parametric curves is regular, because an arbitrary u iso-parametric curve intersects every v iso-parametric curve, and then, there is no tangent point, i.e. non-regular point, between the two sets of iso-parametric curves. By the regular parametric curve grid, the mesh surface S can be mapped to the unit square parameter field $[0,1]\times[0,1]$, while the four pieces of boundaries of the surface S are mapped to the four boundaries of the unit square field.

3.3 Finite element method for solving variational problem

The solution to Eq. (2) is generally obtained by solving the variational problem (3) using the finite element method in engineering[23]. In what follows, we elaborate the method for solving the variational problem (3) step by step.

1) Tetrahedral triangulation. Suppose the heat conductor V is sampled by $n+m$ points, where n points are sampled in the interior G, numbered from 1 to n, and m points sampled on the boundary B, numbered from $n+1$ to $n+m$. A simplicial complex composed of r tetrahedra $V_h(h=1,2,\cdots,r)$ is calculated by tetrahedral triangulation of the sampled points.

2) Piecewise linear function construction. The temperature value is supposed to be linear within each tetrahedron. That is, for the h-th tetrahedron V_h with four vertices $\boldsymbol{P}_i(x_i, y_i, z_i)$, $\boldsymbol{P}_j(x_j, y_j, z_j)$, $\boldsymbol{P}_k(x_k, y_k, z_k)$, and $\boldsymbol{P}_l(x_l, y_l, z_l)$, with temperature values $T_i=T(x_i, y_i, z_i)$, $T_j=T(x_j, y_j, z_j)$, $T_k=T(x_k, y_k, z_k)$, and $T_l=T(x_l, y_l, z_l)$, respectively, the temperature value at any point $\boldsymbol{P}(x, y, z)$ in V_h is

$$T^{(h)}(x,y,z)=\lambda_i(x,y,z)T_i+\lambda_j(x,y,z)T_j+\lambda_k(x,y,z)T_k+\lambda_l(x,y,z)T_l, \tag{9}$$

where λ_i, λ_j, λ_k and λ_l are the barycentric coordinates of $\boldsymbol{P}(x,y,z)$ with respect to V_h.

3) Element stiffness matrix for each tetrahedron. For the h-th tetrahedron V_h, from

$$\frac{\partial T^{(h)}}{\partial t}=T_i\frac{\partial\lambda_i}{\partial t}+T_j\frac{\partial\lambda_j}{\partial t}+T_k\frac{\partial\lambda_k}{\partial t}+T_l\frac{\partial\lambda_l}{\partial t},\quad t=x,y,z, \tag{10}$$

we have

$$\begin{aligned}\boldsymbol{I}^{(h)}&=\iiint_{V_h}\left[\left(\frac{\partial T^{(h)}}{\partial x}\right)^2+\left(\frac{\partial T^{(h)}}{\partial y}\right)^2+\left(\frac{\partial T^{(h)}}{\partial z}\right)^2\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &=[T_i\quad T_j\quad T_k\quad T_l]\left\{\|V_h\|\sum_{t=x,y,z}\begin{bmatrix}\left(\frac{\partial\lambda_i}{\partial t}\right)^2 & \frac{\partial\lambda_i}{\partial t}\frac{\partial\lambda_j}{\partial t} & \frac{\partial\lambda_i}{\partial t}\frac{\partial\lambda_k}{\partial t} & \frac{\partial\lambda_i}{\partial t}\frac{\partial\lambda_l}{\partial t}\\ \frac{\partial\lambda_j}{\partial t}\frac{\partial\lambda_i}{\partial t} & \left(\frac{\partial\lambda_j}{\partial t}\right)^2 & \frac{\partial\lambda_j}{\partial t}\frac{\partial\lambda_k}{\partial t} & \frac{\partial\lambda_j}{\partial t}\frac{\partial\lambda_l}{\partial t}\\ \frac{\partial\lambda_k}{\partial t}\frac{\partial\lambda_i}{\partial t} & \frac{\partial\lambda_k}{\partial t}\frac{\partial\lambda_j}{\partial t} & \left(\frac{\partial\lambda_k}{\partial t}\right)^2 & \frac{\partial\lambda_k}{\partial t}\frac{\partial\lambda_l}{\partial t}\\ \frac{\partial\lambda_l}{\partial t}\frac{\partial\lambda_i}{\partial t} & \frac{\partial\lambda_l}{\partial t}\frac{\partial\lambda_j}{\partial t} & \frac{\partial\lambda_l}{\partial t}\frac{\partial\lambda_k}{\partial t} & \left(\frac{\partial\lambda_l}{\partial t}\right)^2\end{bmatrix}\right\}\begin{bmatrix}T_i\\ T_j\\ T_k\\ T_l\end{bmatrix},\end{aligned} \tag{11}$$

where $\|V_h\|$ denotes the volume of the tetrahedron V_h. The matrix in the brackets of Eq. (11) is called the element stiffness matrix of the tetrahedron V_h.

4) Global stiffness matrix construction. From Eq. (11),

$$I(T)=\sum_{h=1}^{r}\boldsymbol{I}^{(h)}=\boldsymbol{T}'\boldsymbol{K}\boldsymbol{T}, \tag{12}$$

where $\boldsymbol{T}'=[T_1\quad\cdots\quad T_n\quad T_{n+1}\quad\cdots\quad T_{n+m}]$. The matrix $\boldsymbol{K}_{(n+m)\times(n+m)}$ in Eq. (12) is called the global stiffness matrix. It is combined by the element stiffness matrices of all tetrahedra according to the suffix of their elements. That is, the element e in some element stiffness matrix is added to the global matrix with row and column specified by e' s suffix. For example, the element $\|V_h\|\sum_{t=x,y,z}\frac{\partial\lambda_i}{\partial t}\frac{\partial\lambda_j}{\partial t}$ of the element stiffness matrix in Eq. (11) is added to the i-th row and j-th column of the global matrix. Eq. (12) is a multivariable quadratic function with variables T_1, T_2, $\cdots$, T_{n+m}. Therefore, we can get a linear system

$$\boldsymbol{K}\boldsymbol{T}=\boldsymbol{0}, \tag{13}$$

whose solution T_1, T_2, $\cdots$, T_{n+m} are extremum of $I(T)$.

5) Boundary conditions. In Eq. (2), the temperature value on the boundary has been given as $T|_B$

$=\varphi(x, y, z)$, so we need to delete the last m equations in the system of equations (Eq. (13)), i.e. the equations from the $(n+1)$-th to $(n+m)$-th. The terms containing T_{n+1}, T_{n+2}, $\cdots$, T_{n+m} are regarded as constant and are moved to the right of the remaining equations. Thus, we get the following linear system

$$\sum_{j=1}^{n} k_{ij} T_j = G_i, \quad i = 1, 2, \cdots, n. \tag{14}$$

Because the matrix $[k_{ij}]_{n\times n}$ is a positive definite matrix, the system of equations (Eq. (14)) has a unique solution.

4 Experiments and results

In this section, the parametric curve grids on some triangular mesh surfaces are shown in Figs. 2—6, where the u-curves (red curves) are the isotherms in the steady temperature field on the surface, and v-curves (green curves) are obtained by calculating the normalized chord length of the points at the u-curves. Fig. 2 shows the parameter curve grid on the smooth triangular mesh surface head-mask (cut from the model mannequin). Fig. 3 illustrates the mesh of the model ear and different views of the parameter grid on the mesh surface. In Fig. 4, there is the mesh surface cut from the model fan-disk which has a sharp edge, and the parameter curve grid on it. There are different views of the parameter grid on the mesh surface Stanford bunny in Fig. 5. Finally, Fig. 6 shows different views of the parameter grid on the coarse mesh surface cut from the model skull. Illustrated by the above results, whatever the mesh surface is coarse or smooth, the distribution of the iso-parameter curves of the parameter grid generated by the parameterization method based on steady temperature field is uniform. Furthermore, the parameter grid is nearly orthogonal, which is ideal for fitting the mesh surface with NURBS surface. In addition, experimental results for parameterization algorithm based on steady temperature field are listed in Table 1. All the examples run on the PC with 1.8 GHz CPU and 512 MB memory.

Table 1. Experimental results for parameterization algorithm based on steady temperature

Name	Vertices	Faces	Run time(s)
Head-mask	954	1852	1. 0115
Ear-cut	1199	2352	1. 3720
Fan-disk	1676	3175	1. 8430
Bunny	9581	19029	78. 1070
Skull	18409	36534	335. 5275

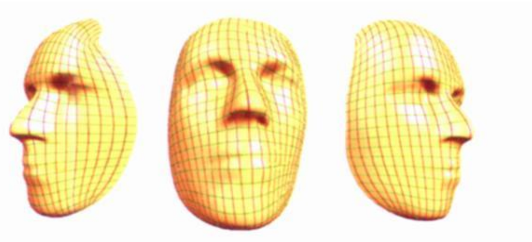

Fig. 2. Parametric curve grid on a triangular mesh surface head-mask. Left: one side view; middle: front view; right: the other side view.

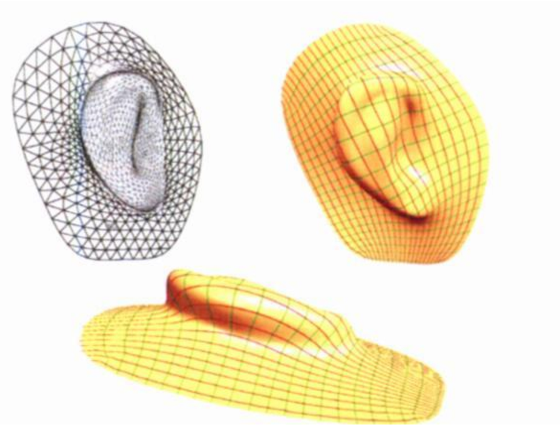

Fig. 3. Mesh surface ear-cut and parametric curve grid on the mesh surface ear-cut. Top left: triangular mesh; top right: front view of the parametric grid; bottom: top view of the parametric grid.

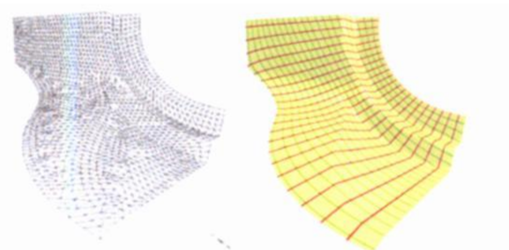

Fig. 4. Mesh surface fan-disk and parametric curve grid on the mesh surface. Left: triangular mesh fan-disk; right: front view of the parametric grid.

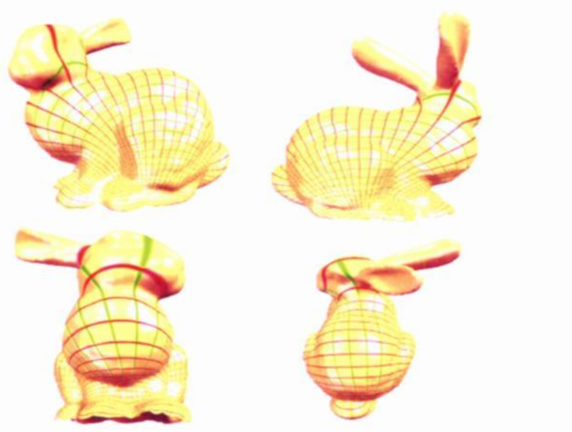

Fig. 5. Parametric curve grid on the mesh surface bunny. Top left: one side view of the parametric grid; top right: the other side view; bottom left: front view; bottom right: top view.

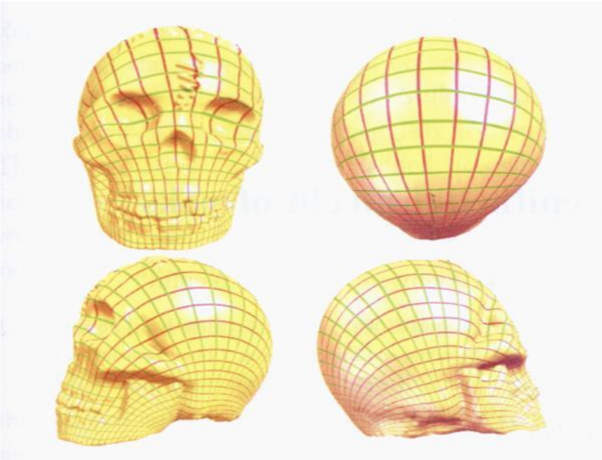

Fig. 6. Parametric curve grid on the mesh surface skull. Top left: front view of the parametric grid; top right: back view; bottom left: one side view; bottom right: the other side view.

5 Conclusions

In reverse engineering, the triangular mesh surface needs to be fitted with NURBS surface. It firstly needs to find a regular parametric curve grid with uniform distribution on the triangular mesh surface. In this paper, we have presented a novel method for parameterizing an arbitrary triangular mesh with four boundaries. Specifically, the steady temperature field on the mesh surface is firstly calculated; and then, the isotherms of the steady temperature field are taken as a set of iso-parametric curves; finally, the other set of iso-parametric curves is obtained by calculating the normalized chord length at the points of the isotherms. The two sets of iso-parametric curves construct the parametric curve grid on the mesh surface, and parameterize the mesh vertices into the unit square. Illustrations show that the distribution of the iso-parametric curves is uniform, and the parametric curve grid is nearly orthogonal, which is important for fitting the mesh surface with NURBS surface. Future work will study the influence of the heat conductivity coefficient on the distribution of the parametric curve grid on the triangular mesh surface.

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